The Derivative

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Introduction

This short paper clarifies several definitions of "the derivative" one encounters in introductory vector calculus.

General Definitions

The derivative of a single function of n variables is a linear transformation from $\Re^n \to \Re^1$. It maps each point in $\Re^n$ onto the "slope" of the n-dimensional "plane" tangent at that point. When several functions are defined (say, m of them), the derivative is a linear transformation from $\Re^n \to \Re^m$ that maps each point in $\Re^n$ onto the "slopes" of the m n-dimensional tangent "planes". One "plane" is tangent to each function.

The Derivative of a Vector of k Functions of One Variable

Define $\vec{x}$ as a vector of functions of one variable, t. That is,

$$\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \end{pmatrix}$$

The derivative of a vector $\vec{x}$, which can be written several ways, is

$$\mathrm{D}\vec{x} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{pmatrix} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}x_k(t)}{\mathrm{d}t} \end{pmatrix}$$

The Gradient

Define a function f of several variables $x_1, x_2, \ldots, x_k$ as $f(x_1, x_2, \ldots, x_k)$. The gradient of f, $\nabla f(x_1, x_2, \ldots, x_k)$, is the vector of partial derivatives of f. That is,

$$\nabla f(x_1, x_2, \ldots, x_k) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_k} \right)$$

The Jacobian

Define a set of m functions $f_1, f_2, \ldots, f_m$. Each of these is a function of several variables. In particular, define the f_i as functions of n variables $x_1, x_2, \ldots, x_n$. The Jacobian is the matrix of all n partial derivatives of all m functions. That is,

$$\mathrm{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_k} \end{pmatrix}$$

Other Names

In some contexts, the Jacobian is called the "total derivative" (Bressoud, 1991), or the "derivative matrix" (Marsden, Tromba and Weinstein, 1993)

References

Bressoud, David M. 1991. *Second-Year Calculus.* New York: Springer-Verlag.

Marsden, Jerrold E., Anthony J. Tromba and Alan Weinstein. 1993. *Basic Multivariable Calculus.* New York: Springer-Verlag.